Chapter 15 Practice Exercises.

13. Let $2x+4=4-x^2$, we obtain $x=0$ and $x=-2$.

The area $= \int_{-2}^{0}\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^{0}(-x^2-2x)\,dx = 4/3$

21. $(x^2+y^2)^2-(x^2-y^2)=0 \xRightarrow[y=r\sin\theta]{x=r\cos\theta} r^4-r^2\cos 2\theta \Rightarrow r^2=\cos 2\theta$ so the integral is

$$\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta + \int_{3\pi/4}^{5\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta = 2\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$$

$$= 2\int_{-\pi/4}^{\pi/4} -\frac{1}{2}\cdot\frac{1}{1+r^2}\Big|_0^{\sqrt{\cos 2\theta}} d\theta = \int_{-\pi/4}^{\pi/4} 1-\frac{1}{1+\cos 2\theta}\,d\theta = \int_{-\pi/4}^{\pi/4} 1-\frac{1}{2\cos^2\theta}\,d\theta = \left(\theta-\frac{\tan\theta}{2}\right)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{\pi-2}{2}$$

28. $V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-x^2)\,dy\,dx = 2\int_{-2}^{2}(4-x^2)^{3/2}\,dx$

$$\overset{x=2\cos\theta}{=} 2\int_{\pi}^{0}(4\sin^2\theta)^{\frac{3}{2}}\,d(2\cos\theta) = 32\int_0^{\pi}\sin^4\theta\,d\theta = 32\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= 32\int_0^{\pi}\left[\frac{1}{4}-\frac{1}{2}\cos 2\theta+\frac{1}{8}(1+\cos 4\theta)\right]d\theta = 32\times\frac{3}{8}\times\pi = 12\pi$$

32. $\begin{cases} 0\le x\le 1 \\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2} \\ -(x^2+y^2)\le z\le x^2+y^2 \end{cases} \Rightarrow \begin{cases} 0\le r\cos\theta\le 1 \\ r^2\sin^2\theta\le 1-r^2\cos^2\theta \\ -r^2\le z\le r^2 \end{cases} \Rightarrow \begin{cases} -\pi/2\le\theta\le\pi/2 \\ 0\le r\le 1 \\ -r^2\le z\le r^2 \end{cases}$

(a) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2\,dz\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^4\cos\theta\sin^2\theta\,dz\,dr\,d\theta$

(b) $= 84\int_0^{\pi/2}\int_0^1 r^6\sin^2\theta\cos\theta\,dr\,d\theta = 12\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta = 4$

33. $\begin{cases} -1\le x\le 1 \\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2} \\ \sqrt{x^2+y^2}\le z\le 1 \end{cases} \Rightarrow \begin{cases} \rho^2\sin^2\phi\cos^2\theta\le 1 \\ \rho^2\sin^2\phi\sin^2\theta\le 1-\rho^2\sin^2\phi\cos^2\theta \\ \rho\sin\phi\le\rho\cos\phi\le 1 \end{cases} \Rightarrow \begin{cases} \rho^2\sin^2\phi\cos^2\theta\le 1 \\ \rho^2\sin^2\phi\le 1 \\ \tan\phi\le 1,\ \rho\le\sec\phi \end{cases}$

$\Rightarrow \begin{cases} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi/4 \\ 0\le\rho\le\sec\phi \end{cases}$

(a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sec\phi(\sec\phi\tan\phi)\,d\phi\,d\theta$

$$= \frac{1}{6}\int_0^{2\pi}\tan^2\phi\Big|_0^{\pi/4}d\theta = \frac{1}{6}\int_0^{2\pi}d\theta = \frac{\pi}{3}$$

53. $\begin{cases} u = x-y \\ v = y \end{cases} \Rightarrow \begin{cases} x = u+v \\ y = v \end{cases}$ $\quad \begin{cases} 0<x<+\infty \\ 0<y<x \end{cases} \Rightarrow \begin{cases} 0<u+v<+\infty \\ 0<v<u+v \end{cases} \Rightarrow \begin{cases} 0<u<+\infty \\ 0<v<+\infty \end{cases}$

$J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow \int_0^\infty \int_0^x e^{-sx} f(x-y,y)\,dy\,dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u,v)\,du\,dv$

Chapter 15 Additional and Advanced Exercise

11. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \int_0^\infty \int_a^b e^{-xy}\,dy\,dx = \int_a^b \int_0^\infty e^{-xy}\,dx\,dy = \int_a^b -\frac{e^{-xy}}{y}\Big|_0^\infty dy$

$= \int_a^b \frac{1}{y}\,dy \ (b>a>0) = \ln\frac{b}{a}$

19. $\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\,dy\,dx = \int_0^a \int_0^{bx/a} e^{b^2x^2}\,dy\,dx + \int_0^b \int_0^{ay/b} e^{a^2y^2}\,dx\,dy$

$= \int_0^a \frac{b}{a} x e^{b^2x^2}\,dx + \int_0^b \frac{a}{b} y e^{a^2y^2}\,dy = \frac{1}{2ab} e^{b^2x^2}\Big|_0^a + \frac{1}{2ba} e^{a^2y^2}\Big|_0^b = \frac{1}{ab}\left(e^{a^2b^2}-1\right)$